An application of Kapteyn series to a problem from queueing theory

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1 Introduction

In [2, 4.47], the authors considered the following problem:

Problem 1 *Find the unique solution $C(D)$ in the range $(0,1)$ of the transcendental equation*

$$1=\frac{D}{2\left(D+1\right)}F(C), \tag{1}$$

where

$$F(C)=\sum_{n=-\infty}^{\infty}J_{n}\left(\frac{n}{\sqrt{D+1}}\right)C^{n}, \tag{2}$$

$J_{n}(\cdot)$ is the Bessel function of the first kind and $D>0$.

They also defined the functions $C_{1}\left(D,a\right)$ and $C_{2}\left(D,a\right)$ implicitly by

$$0=\frac{a}{2\sqrt{D+1}}F(C)+\frac{1}{2}C_{1}F_{1}(C), \tag{3}$$

$$0=\frac{1}{4\sqrt{D+1}}\left[1-\frac{a^{2}}{2\left(D+1\right)}\right]F_{2}(C)+\left[\frac{1}{4}\frac{a}{\sqrt{D+1}}C_{1}+C_{2}\right]F_{1}(C), \tag{4}$$

where $a>0$ and

$$F_{1}(C)=\sum_{n=-\infty}^{\infty}nJ_{n}\left(\frac{n}{\sqrt{D+1}}\right)C^{n},\quad F_{2}(C)=\sum_{n=-\infty}^{\infty}nJ_{n}^{\prime}\left(\frac{n}{\sqrt{D+1}}\right)C^{n}. \tag{5}$$

Using some properties and asymptotic approximations for the Bessel functions, they proved that

$$\left(\sqrt{D+1}-\sqrt{D}\right)\exp\left(\sqrt{\frac{D}{D+1}}\right)<C(D)<1, \tag{6}$$

$$C(D)\sim1-\frac{D^{2}}{4},\quad C_{1}(D)\sim\frac{aD}{2},\quad C_{2}(D)\sim\frac{1}{4}-\frac{a^{2}}{8},\quad D\rightarrow0,\quad C(D)\sim\sqrt{\frac{e}{D}},\quad D\rightarrow\infty. \tag{7}$$

Series of the form (2), (5) are called Kapteyn series [1]. The purpose of this work is to obtain exact solutions of (1), (3) and (4) using properties of such series.

2 Main Result

Theorem 2 *The solutions of (1), (3) and (4) are given by*

$$C(D)=\frac{\exp\left(\frac{1}{2}\frac{D}{D+1}\right)}{\sqrt{D+1}},\quad C_{1}(D)=\frac{D}{2\left(D+1\right)^{\frac{3}{2}}}a,\quad C_{2}(D)=\frac{\sqrt{D+1}}{4}-\frac{D+\left(D+1\right)^{\frac{3}{2}}}{8\left(D+1\right)^{2}}a^{2}. \tag{8}$$

Proof. Let

$$\varepsilon = \frac{1}{\sqrt{D+1}}, \quad C = e^{\mathrm{i}M}, \quad M = E - \varepsilon \sin(E). \tag{9}$$

Using (9) and the formula [3, 2.1 (2)],

$$J_{-n}(z) = (-1)^n J_n(z), \tag{10}$$

we can rewrite (1) as

$$\frac{2}{1-\varepsilon^2} = 1 + 2\sum_{n=1}^{\infty} J_n(n\varepsilon)\cos(nM) = \frac{1}{1-\varepsilon\cos(E)}, \tag{11}$$

where we have used [3, 17.21 (6)] and

$$r = \frac{\alpha\left(1-\varepsilon^2\right)}{1+\varepsilon\cos(\omega)} = \alpha\left[1-\varepsilon\cos(E)\right]. \tag{12}$$

It follows from (11) that

$$E = \arccos\left(\frac{1+\varepsilon^2}{2\varepsilon}\right) \tag{13}$$

and therefore

$$M = \arccos\left(\frac{1+\varepsilon^2}{2\varepsilon}\right) + \frac{\mathrm{i}}{2}\left(\varepsilon^2-1\right).$$

Thus,

$$C(D) = e^{\mathrm{i}M} = \varepsilon\exp\left(\frac{1-\varepsilon^2}{2}\right) = \frac{\exp\left(\frac{1}{2}\frac{D}{D+1}\right)}{\sqrt{D+1}}.$$

Using (9) and (10) in (5), we have [3, 17.21 (9-10)]

$$F_1(C) = 2\mathrm{i}\sum_{n=1}^{\infty} nJ_n(n\varepsilon)\sin(nM) = \frac{\alpha^2}{r^2}\sin(\omega)\frac{\varepsilon}{\sqrt{1-\varepsilon^2}}\mathrm{i}, \tag{14}$$

$$F_2(C) = 2\sum_{n=1}^{\infty} nJ_n'(n\varepsilon)\cos(nM) = \frac{\alpha^2}{r^2}\cos(\omega),$$

where [3, 17.2 (1-3)]

$$\sin(\omega) = \frac{\sqrt{1-\varepsilon^2}}{1-\varepsilon\cos(E)}\sin(E). \tag{15}$$

Using (12), (13) and (15) in (14), we obtain

$$F_1(C) = -\frac{4}{\left(1-\varepsilon^2\right)^2} = -4\left(\frac{D+1}{D}\right)^2, \quad F_2(C) = \frac{4}{\varepsilon\left(1-\varepsilon^2\right)^2} = 4\frac{(D+1)^{\frac{5}{2}}}{D^2}. \tag{16}$$

Replacing (16) in (3)-(4), the result follows. ■

An easy computation, shows that (8) implies (6) and (7).

References

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